International Journal of Theoretical Physics, Vol. 20, No. 5, 1981

Gravitational Field of a Sphere Composed of Concentric Shells

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Received June 5, 1980

A general solution of the field equations of general relativity theory has been obtained for a composite sphere having a number of concentric shells of different densities.

1. INTRODUCTION

We have some particular solutions (Tolman, 1939; Wyman, 1949; Buchdal, 1964; Kuchowicz, 1966; Mehra, 1966; Adler, 1974) which describe the interior gravitational field of spherically symmetric static bodies having variable densities. In general relativity it is very difficult to get an analytic solution of the field equations for any kind of variable density. Mehra (1968) and Mehra, Vaidya, and Kushwaha (1969) therefore have given the simple device to describe the interior gravitational field of bodies having any kind of variable densities. Accordingly, the body may be considered as a composite sphere having a number of concentric shells, one above the other, of different densities. The number of shells and their densities may be taken according to the distribution of matter in the body. This device has also been used by Bohra and Mehra (1971), Durgapal and Gehlot (1969), Durgapal (1971, 1972), Gehlot and Durgapal (1971), Durgapal (1974), Krori (1970, 1971), and Krori and Borgohain (1974). They have assumed two or three density distributions in the bodies.

In this paper the same device has been used to obtain another general solution of the field equations to describe the interior gravitational field of spherically symmetrical static bodies having any kind of variable density. Here we consider different variable densities in shells. The following general

0020-7748/81/0500-0377$03.00/0

assumptions are made here for solving the field equations:

(a) The constant density ρ_0 is in the core having radius R_0 and variable densities $K_1/r^2, K_2/r^2, \ldots, K_n/r^2$ are in n shells having outer radii $R_1, R_2, \ldots, R_n$, respectively. Here $K_1, K_2, \ldots, K_n$ are constants which must satisfy the following condition:

$$K_1 > K_2 > K_3 > \cdots > K_n$$

(b) The space is empty outside the radius R_n, i.e., the gravitational field outside the body is described by the well-known Schwarzschild exterior solution.

(c) The gravitational potentials e^λ and e^ν must be continuous everywhere.

(d) The pressure must be positive, finite, and continuous everywhere inside the body and zero at and outside the surface of the body.

2. FIELD EQUATIONS AND THEIR SOLUTIONS

The line element is given by

$$ds^2 = e^{\nu(r)}\,dt^2 - e^{\lambda(r)}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 \tag{1}$$

The resulting field equations for perfect fluid at rest are given by Tolman (1962) as

$$8\pi p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{2}$$

$$8\pi p = e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right) \tag{3}$$

$$8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{4}$$

where the prime denotes differentiation with respect to r.

The solutions of the field equations (2)–(4) for the core, shells (Bohra and Mehra, 1971; Kuchowicz 1966), and outside the body are given below.

For core $0 \leqslant r < R_0$,

$$e^{-\lambda} = 1 - 8\pi r^2\rho_0/3 \tag{5a}$$

$$e^\nu = \left|A_0 - B_0\left(1 - 8\pi r^2\rho_0/3\right)^{1/2}\right|^2 \tag{5b}$$

$$p = \frac{\rho_0}{3}\left|\frac{3B_0\left(1 - 8\pi r^2\rho_0/3\right)^{1/2} - A_0}{A_0 - B_0\left(1 - 8\pi r^2\rho_0/3\right)^{1/2}}\right| \tag{5c}$$

For shells $R_i < r < R_{i+1}$,

$$e^{-\lambda} = 1 - 8\pi K_i + C_i/r, \qquad i = 1,2,3,\ldots,n \tag{6a}$$

$$e^{\nu} = \left(A_i F_i + B_i X_i^{5/2} E_i\right)^2 / X_i^2 \tag{6b}$$

$$8\pi p = C_i(1-X_i)\left(\frac{8\pi K_i - 1}{C_i X_i}\right)^3 \left| \frac{A_i\left(2X_i\bar{F}_i - F_i\right) + 2B_i X_i^{5/2}\left(2E_i + X_i\bar{E}_i\right)}{A_i F_i + B_i X_i^{5/2} E_i} - \frac{X_i}{(8\pi K_i - 1)(1-X_i)} \right| \tag{6c}$$

where

$X_i = r(8\pi K_i - 1)/C_i$

$F_i =$ hypergeometric function $F\left(-1 + (1 - 8\pi K_i)^{-1/2}, -1 - (1 - 8\pi K_i)^{-1/2}; -3/2; X_i\right)$

$E_i =$ hypergeometric function $F\left(3/2 + (1 - 8\pi K_i)^{-1/2}, 3/2 - (1 - 8\pi K_i)^{-1/2}; 7/2; X_i\right)$

$$\bar{F}_i = \frac{dF_i}{dX_i}$$

$$\bar{E}_i = \frac{dE_i}{dX_i}$$

For outside $R_n < r \leqslant \infty$,

$$e^{-\lambda} = 1 - 2M/r \tag{7a}$$

$$e^{\nu} = 1 - 2M/r \tag{7b}$$

$$p = 0 \tag{7c}$$

In the above solutions (5), (6), and (7), A_0, B_0, A_i, B_i, C_i, and M are $3n+3$ constants of integration. To determine these constants we have equal number of conditions of continuity of e^{λ}, e^{ν}, and p at $r = R_0, R_1, R_2, \ldots, R_n$

3. DETERMINATION OF CONSTANTS

Applying the continuity of e^{λ} at $r=R_0, R_1, R_2, \ldots, R_n$ successively, we have

$$C_1 = 8\pi R_0\left(3K_1 - R_0^2\rho_0\right)/3 \tag{8}$$

$$C_j = 8\pi R_{j-1}\left(K_j - K_{j-1}\right) + C_{j-1}, \qquad j=2,3,\ldots,n \tag{9}$$

$$\begin{aligned} M &= 4\pi R_n K_n - C_n/2 \\ &= 4\pi\left|R_n K_n + R_{n-1}\left(K_{n-1} - K_n\right) + \cdots + R_0\left(R_0^2\rho_0/3 - K_1\right)\right| \end{aligned} \tag{10}$$

The constant M is, therefore, identifiable as mass of the body. The continuity of e^{ν} and p at $r=R_n$ gives

$$\begin{aligned} A_n = (1-2M/R_n)^{1/2}\Big|X_n\Big\{D_n - 2X_n\bar{F}_n E_n \\ - E_n F_n\left(R_n/C_n(1-X_n)+1\right)\Big\}/F_n D_n\Big|_{r=R_n} \end{aligned} \tag{11}$$

$$\begin{aligned} B_n = (1-2M/R_n)^{1/2}\Big|\Big\{2X_n\bar{F}_n - F_n\big(R_n/C_n(1-X_n) \\ +1\big)\Big\}/X^{3/2}D_n\Big|_{r=R_n} \end{aligned} \tag{12}$$

where

$$D_i = 2X_i\left(\bar{F}_i E_i - \bar{E}_i F_i\right) - 5E_i F_i$$

On applying both continuities of e^{ν} and p at $r=R_{n-1}, R_{n-2}, \ldots, R_1, R_0$ successively, we have

$$\begin{aligned} A_{s-1} = \Big|X_{s-1}\Big\{A_s\left(2\bar{F}_s E_{s-1}X_s - 2F_s\bar{E}_{s-1}X_{s-1} - 5F_s E_{s-1}\right) - 2B_s X_s^{5/2} \\ \cdot\left(E_s\bar{E}_{s-1}X_{s-1} - \bar{E}_s E_{s-1}X_s\right)\Big\}/X_s D_{s-1}\Big|_{r=R_{s-1}} \end{aligned} \tag{13}$$

$$\begin{aligned} B_{s-1} = \Big|\Big\{B_s X_s^{5/2}\left(2E_s\bar{F}_{s-1}X_{s-1} - 2X_s\bar{E}_s F_{s-1} - 5F_{s-1}E_s\right) \\ + 2A_s\left(F_s\bar{F}_{s-1}X_{s-1} - F_{s-1}\bar{F}_s X_s\right)\Big\}/X_s X_{s-1}^{3/2} D_{s-1}\Big|_{r=R_{s-1}} \end{aligned} \tag{14}$$

$$s = n-1, n-2, n-3, \ldots, 1$$

$$A_0 = \Big|\big(3-8\pi R_0^2\rho_0\big)\big\{2A_1\big(\bar{F}_1X_1-F_1\big)+B_1X_1^{5/2}\big(3E_1+2\bar{E}_1X_1\big)\big\}$$
$$/16\pi R_0^2\rho_0X_1+\big(A_1F_1+B_1X_1^{5/2}E_1\big)/X_1\Big|_{r=R_0} \tag{15}$$

$$B_0 = \Big|3^{1/2}\big(3-8\pi R_0^2\rho_0\big)^{1/2}\big\{2A_1\big(\bar{F}_1X_1-F_1\big)$$
$$+B_1X_1^{5/2}\big(3E_1+2\bar{E}_1X_1\big)\big\}/16\pi R_0^2\rho_0X_1\Big|_{r=R_0} \tag{16}$$

Thus we have determined the values of all the constants of integration. Hence e^λ, e^ν, and p are continuous everywhere, which is necessary for the solution of physical significance.

To obtain the positive pressure at the center which does not exceed $1/3$ of the density at the center, we must have

$$2B_0 < A_0 < 3B_0 \tag{17}$$

If the pressure at the center is positive and does not exceed the density at the center, then we must have

$$3B_0/2 < A_0 < 3B_0 \tag{18}$$

On satisfying the condition (18), the signals cannot propagate at a velocity greater than the velocity of light (Zeldovich, 1961).

4. DISCUSSION

Here we have obtained the general solution to describe the interior gravitational field of spherically symmetric body. Its particular solutions can be used to arrive at astronomical and cosmological facts of the heavenly bodies. The particular solution for $n=1$ has been discussed by Gehlot and Durgapal (1971) and Krori (1971).

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